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INEQUALITIES OF SOLUTIONS OF VOLTERRA INTEGRAL AND DIFFERENTIAL EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper, we study solutions of Volterra integral and differential equations,

$$x'(t) = -a(t)x(t) + \int_{t-h}^{t} b(s)x(s)ds + f(t, x_t), \quad x \in \mathbf{R},$$

or

$$X(t) = a(t) + \int_{t-\alpha}^{t} g(t,s)X(s)ds, \quad X \in \mathbf{R^n}.$$

With Lyapunov functionals, we obtain inequalities for the solutions of these equations. As a corollary, we also obtain a result on asymptotic stability which is simpler and better than some existing results.

Key words and phrases: Differential and integral inequalities, stability, boundedness, functional differential equations.
AMS (MOS) Subject Classifications: 34A40, 34K20

1 Introduction

Before proceeding, we shall set forth notation and terminology that will be used throughout this paper. Let $A = (a_{ij})$ be an $n \times n$ matrix. A^T denotes the transpose of A, $A^T = (a_{ji})$, and $|A| = \sqrt{\Sigma_{i,j=1}^{n} a_{ij}^2}$. Let $(C, ||\cdot||)$ be the Banach space of continuous functions $\phi : [-h, 0] \to \mathbf{R^n}$ with the norm $||\phi|| = max_{-h \le s \le 0}|\phi(s)|$ and $|\cdot|$ is any convenient norm in $\mathbf{R^n}$. In this paper, we will use the norm defined by $|X| = \sqrt{\Sigma_{i=1}^{n} x_i^2}$ for $X = (x_1, x_2, ..., x_n)^T \in \mathbf{R^n}$. Given $H > 0$, by C_H we denote the subset of $\mathbf{C}$ for which $||\phi|| < H$. $X'(t)$ denotes the right-hand derivative at t if it exists and is finite. Definitions of stability and boundedness can be found in [1].

2 Some Results on Inequalities of Solutions of Functional Differential Equations

There have been a lot of discussions on estimating solutions of differential equations. For the system of ordinary differential equations

$$X'(t) = A(t)X(t), \quad X \in \mathbf{R}^{\mathbf{n}}, \tag{1}$$

where A is an $n \times n$ real matrix of continuous functions defined on $\mathbf{R}_+ = [0, \infty)$, solutions are estimated by Ważewski's inequality, which is stated as Theorem 1.1 below and its proof can be found in [3, 12].

Theorem 2.1 (Ważewski's inequality) *Consider (1). Let* $H(t) = \frac{1}{2}(A^T(t) + A(t))$ *and* $\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)$ *be the n eigenvalues of* $H(t)$. *Let*

$$\lambda(t) = min\{\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)\}, \quad \Lambda(t) = max\{\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)\}.$$

If $X(t)$ *is a solution of (1), then*

$$|X(t_0)|e^{\int_{t_0}^t \lambda(s)ds} \leq |X(t)| \leq |X(t_0)|e^{\int_{t_0}^t \Lambda(s)ds}.$$

For the nonlinear non-autonomous system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} G_{11}(t,X) & \cdots & G_{1r}(t,X) \\ \vdots & \ddots & \vdots \\ G_{r1}(t,X) & \cdots & G_{rr}(t,X) \end{pmatrix} \begin{pmatrix} F_1(x_1) \\ F_2(x_2) \\ \vdots \\ F_r(x_r) \end{pmatrix},$$

solutions are estimated by Ważewski's type inequalities and details can be found in [4].

For the linear Volterra integro-differential system

$$X'(t) = A(t)X(t) + \int_{t-h}^t B(t,s)X(s)ds + F(t), \quad X \in \mathbf{R}^{\mathbf{n}}, \tag{2}$$

where $h > 0$ is a constant, A is an $n \times n$ real matrix of continuous functions defined on $\mathbf{R}_+ = [0, \infty)$, B is an $n \times n$ real matrix of continuous functions defined on $\{(t,s)| -\infty < s \leq t < \infty\}$, and $F : \mathbf{R}_+ \to \mathbf{R}^{\mathbf{n}}$ is continuous, the following inequalities estimate its solutions [7].

Theorem 2.2 *Consider (2). Let* $\Lambda(t)$ *be as in Theorem 2.1. Assume that there is a* $K > 0$ *such that for each* (t,s), $-\infty < t - h \leq s \leq t < \infty$,

$$|B(t,s)| - K|\Lambda(s)| \leq K(\Lambda(t) + Kh|\Lambda(t)|)|\Lambda(s)|(s - t + h).$$

Denote $\Lambda^*(t) = \Lambda(t) + Kh|\Lambda(t)|$. *If* $X(t) = X(t, t_0, \phi)$ *is a solution of (2), then for* $t \geq t_0$,

$$|X(t)| \leq e^{\int_{t_0}^t \Lambda^*(s)ds}\left[M(t_0) + \int_{t_0}^t |F(s)|e^{-\int_{t_0}^s \Lambda^*(u)du}ds\right],$$

where $M(t_0) = |\phi(0)| + K\int_{-h}^0\int_s^0 |\Lambda(t_0+u)||\phi(u)|\ duds$.

Theorem 2.3 *Consider (2). Let* $\lambda(t)$ *be as in Theorem 1.1. Assume that there is a* $k > 0$ *such that for each* $(t,s), -\infty < t-h \leq s \leq t < \infty$,

$$|B(t,s)| - k|\lambda(s)| \leq k(\lambda(t) - kh|\lambda(t)|)|\lambda(s)|(s-t+h).$$

Denote $\lambda_*(t) = \lambda(t) - kh|\lambda(t)|$. *If* $X(t) = X(t, t_0, \phi)$ *is a solution of (2), then for* $t \geq t_0$

$$|X(t)| \geq e^{\int_{t_0}^t \lambda_*(s)ds}\left[m(t_0) - \int_{t_0}^t |F(s)|e^{-\int_{t_0}^s \lambda_*(u)du}ds\right],$$

where $m(t_0) = |\phi(0)| - k\int_{-h}^0\int_s^0 |\lambda(t_0+u)||\phi(u)|\ duds$.

For the linear scalar functional differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-h), \tag{3}$$

where $a, b : \mathbf{R}_+ \to \mathbf{R}$ continuous, and $h > 0$ is a constant, we obtained the following three inequalities [9].

Theorem 2.4 *Assume* $-\frac{1}{2h} \leq a(t) + b(t+h) \leq -hb^2(t+h)$. *Let* $x(t) = x(t, t_0, \phi)$ *be a solution of (3) defined on* $[t_0, \infty)$. *Then*

$$|x(t)| \leq ||\phi||\left(1 + \int_{t_0}^{t_0+\frac{h}{2}} |b(u)|du\right)e^{\int_{t_0}^t a(s)ds}$$

for $t \in [t_0, t_0 + \frac{h}{2}]$; *and*

$$|x(t)| \leq \sqrt{6V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\frac{h}{2}}[a(s)+b(s+h)]ds}$$

for $t \geq t_0 + \frac{h}{2}$, *where*

$$V(t_0) = [\phi(0) + \int_{-h}^0 b(s+t_0+h)\phi(s)ds]^2 + h\int_{-h}^0 b^2(z+t_0+h)\phi^2(z)dz.$$

Theorem 2.5 *Let* $x(t) = x(t, t_0, \phi)$ *be a solution of Equation(3) defined on* $[t_0, \infty)$. *If there is a constant* $\beta > 0$, *such that* $|b(t)| \leq h\mu(t)$, *where* $\mu(t) = \frac{e^{\int_0^t a(s)ds}}{1+h\int_t^{t+\beta} e^{\int_0^u a(s)ds}du}$, *then*

$$|x(t)| \leq V(t_0)e^{\int_{t_0}^t [a(s)+h\mu(s)]ds}$$

where

$$V(t_0) = |\phi(0)| + h\mu(t_0)\int_{-h}^0 |\varphi(s)|ds.$$

Theorem 2.6 *Let $H > h$ and*

$$a(t) + b(t+h) - Hb^2(t+h) \geq 0.$$

If $x(t) = x(t, t_0, \phi)$ is a solution to (3) defined on $[t_0, \infty)$, then

$$x^2(t) \geq \frac{H-h}{H} V(t_0) e^{\int_{t_0}^{t} [a(s)+b(s+h)]ds}$$

where $V(t_0) = [\phi(0) + \int_{-h}^{0} b(s+h)\phi(s)ds]^2 - H \int_{-h}^{0} b^2(s+h)\phi^2(s)ds$.

For the general abstract functional differential system with finite delay

$$\frac{du}{dt} = F(t, u_t), \quad u_t(s) = u(t+s), \tag{4}$$

we obtained the following results [10].

Theorem 2.7 *Let $V : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous and $D : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous along the solutions of (4), and $\eta, L,$ and $P : \mathbf{R}_+ \to \mathbf{R}_+$ be integrable. Suppose the following conditions hold:*

i) $W_1(|u(t)|_{\mathbf{X}}) \leq V(t, u_t) \leq W_2(D(t, u_t)) + \int_{t-h}^{t} L(s) W_1(|u(s)|_{\mathbf{X}})ds$,

ii) $V'_{(1)}(t, u_t) \leq -\eta(t) W_2(D(t, u_t)) + P(t)$.

Then the solutions of (4), $u(t) = u(t, t_0, \phi)$, satisfy the following inequality:

$$W_1(|u(t)|_{\mathbf{X}}) \leq \left[K + \int_{t_0}^{t} P(s) e^{\int_{t_0}^{s} \eta(r)dr} ds \right] e^{\int_{t_0}^{t} [-\eta(s) + L(s)(e^{\int_{s}^{s+h} \eta(r)dr} - 1)]ds}, t \geq t_0, \tag{5}$$

where $K = V(t_0, \phi) + [e^{\int_{t_0}^{t_0+h} \eta(r)dr} - 1] \int_{-h}^{0} L(s+t_0) W_1(|\phi(s)|_{\mathbf{X}})ds$.

Theorem 2.8 *Let M and c be positive constants, and let $u(t) = u(t, t_0, \phi)$ be a solution of (4). Let $V : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous and $D : \mathbf{R}_+ \times \mathbf{C}_{\mathbf{XH}} \to \mathbf{R}_+$ be continuous along the solutions of (4), and assume the following conditions hold:*

i) $W_1(|u(t)|_{\mathbf{X}}) \leq V(t, u_t) \leq W_2(D(t, u_t)) + M \int_{t-h}^{t} W_1(|u(s)|_{\mathbf{X}})ds$,

ii) $V'_{(1)}(t, u_t) \leq -cW_2(D(t, u_t))$,

iii) $hM < 1$.

Then there is a constant $\varepsilon > 0$ such that the solutions of (4) satisfy the following inequality:

$$W_1(|u(t)|_{\mathbf{X}}) \leq Ke^{-\varepsilon(t-t_0)}, \tag{6}$$

where $K = V(t_0, \phi) + M(e^{ch} - 1) \int_{-h}^{0} W_1(|\phi(s)|_{\mathbf{X}})ds$.

Applying Theorem 2.8 to the following partial functional differential equation,

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = u_{xx}(t,x) + \omega u(t,x) + f(u(t-h,x)), \\ &u(t,0) = u(t,\pi) = 0,\ t \geq 0,\ 0 \leq x \leq \pi,\ f(0) = 0, \end{aligned} \tag{7}$$

with ω a real constant and $f : \mathbf{R} \to \mathbf{R}$ continuous, we obtained the following estimate on its solutions.

Theorem 2.9 *Let $-1 + \omega + L < 0$. Then the solutions of (7) satisfy the following inequality:*

$$|u(t,x)|_{\mathbf{H}_0^1} \leq \sqrt{K} e^{\frac{1}{2}[Le^{2(1-\omega-L)h}+2\omega+L-2](t-t_0)},\ t \geq t_0,$$

where

$$K = |\phi(0)(x)|^2_{\mathbf{H}_0^1} + L\int_{-h}^{0} |\phi(s)(x)|^2_{\mathbf{H}_0^1} ds + L\left[e^{2(1-\omega-L)h} - 1\right] \int_{-h}^{0} |\phi(s)(x)|^2_{\mathbf{H}_0^1} ds.$$

In addition, if $hL < 1$, then there exists an $\varepsilon > 0$ such that

$$|u(t,t_0,\phi)|_{\mathbf{H}_0^1} \leq \sqrt{K} e^{-\varepsilon(t-t_0)},\ t \geq t_0,$$

and hence, the zero solution of (7) is exponential asymptotically stable in $(\mathbf{H}_0^1,\ \mathbf{H}_0^1)$.

3 More Estimates on Volterra Integral and Differential Equations

In this part, we investigate more Volterra integral and differential equations. Our results are new and improve some former results.

Example 3.1 *Consider the scalar equation*

$$x'(t) = -a(t)x(t) + \int_{t-h}^{t} b(s)x(s)ds + f(t,x_t), \tag{8}$$

with $a : \mathbf{R}_+ \to \mathbf{R}_+$ and $b : [-h,\infty) \to R$ continuous, and $f(t,\phi) : \mathbf{R}_+ \times \mathbf{C} \to \mathbf{R}$ continuous.

Theorem 3.1 *Suppose that the following conditions hold.*

i) There exists a continuous function, $P(t) : R_+ \to R_+$ such that $|f(t,\phi)| \leq P(t)$ for $(t,\phi) \in R_+ \times \mathbf{C}$.

ii) There is a constant $\theta > 0$ with $0 < \theta h < 1$ such that $|b(t)| - \theta a(t) \leq 0$.

If $x(t) = x(t, t_0, \phi)$ is a solution to (8) defined on $[t_0, \infty)$, then

$$|x(t)| \leq \left[K + \int_{t_0}^{t} P(s) e^{\int_{t_0}^{s} \eta(r)dr} ds \right] e^{-\int_{t_0}^{t} \eta(s) e^{\int_{s}^{s+h} \eta(r)dr} ds}, t \geq t_0, \tag{9}$$

where $\eta(t) := (1-\theta h)a(t)$, $K = V(t_0, \phi) + [e^{\int_{t_0}^{t_0+h} \eta(r)dr} - 1] \int_{-h}^{0} a(s+t_0)|\phi(s)|ds$ and $V(t_0, \phi) = |\phi(0)| + \int_{-h}^{0} a(u+t_0)|\phi(u)|du$.

Proof. Define

$$V(t, x_t) = |x(t)| + \theta \int_{-h}^{0} \int_{t+s}^{t} a(u)|x(u)|duds.$$

Then

$$|x(t)| \leq V(t, x_t) \leq |x(t)| + \theta h \int_{t-h}^{t} a(u)|x(u)|du$$

and

$$\begin{aligned} V'(t, x_t) &\leq -a(t)|x(t)| + \int_{t-h}^{t} |b(u)||x(u)|du + |f(t, x_t)| \\ &+ \theta h a(t)|x(t)| - \theta \int_{t-h}^{t} a(u)|x(u)|du \\ &= (\theta h - 1)a(t)|x(t)| + \int_{t-h}^{t} [|b(u)| - \theta a(u)]|x(u)|du + |f(t, x_t)| \\ &\leq (\theta h - 1)a(t)|x(t)| + P(t). \end{aligned} \tag{10}$$

In Theorem 2.7, take $\eta(t) = (1-\theta h)a(t), L(t) = \theta h a(t)$. By Theorem 2.7, we obtain (10).

Many authors have studied (8). Wang [11] gave results on uniform boundedness and ultimately uniform boundedness. Here we give an estimate for solutions with simpler conditions. For $f(t, x_t) = 0$, Burton, Casal and Somolinos [2] and Wang [5, 6] studied asymptotic stability, uniform stability and uniformly asymptotic stability. In the following theorem, we obtain asymptotic stability with weaker and simpler conditions and an estimate for the solutions. Its proof is a direct corollary of Theorem 3.1.

Theorem 3.2 *Let $f(t, x_t) = 0$ in (8). Suppose that there is a constant $\theta > 0$ with $0 < \theta h < 1$ such that $|b(t)| - \theta a(t) \leq 0$. If $x(t) = x(t, t_0, \phi)$ is a solution of (8) defined on $[t_0, \infty)$, then*

$$|x(t)| \leq K e^{-\int_{t_0}^{t} \eta(s)ds}, \ t \geq t_0, \tag{11}$$

where $\eta(t) := (1-\theta h)a(t)$, $K = V(t_0, \phi) + [e^{\int_{t_0}^{t_0+h} \eta(r)dr} - 1] \int_{-h}^{0} a(s+t_0)|\phi(s)|ds$ and $V(t_0, \phi) = |\phi(0)| + \int_{-h}^{0} a(u+t_0)|\phi(u)|du$. In addition, if $a(t) \notin L^1[0, \infty)$, then $x = 0$ is asymptotically stable.

Let us consider the following integral equation:

$$X(t) = a(t) + \int_{t-\alpha}^{t} g(t,s)X(s)ds, \quad X \in \mathbf{R^n}. \tag{12}$$

Wang [8] obtained uniform boundedness and ultimate uniform boundedness. We will give an estimate of its solutions.

Theorem 3.3 *Let $a(t) \in \mathbf{R^n}$ be continuous on $\mathbf{R}$ and $g(t,s)$ be an $n \times n$ real matrix of continuous functions on $-\infty < s \le t < \infty$. Assume that*

i) $p(t) := |g(t,t)| - \frac{1}{\alpha} \le 0$ for each $t \ge 0$,

ii) $D_r|g(t,s)| \le 0$ for (t,s), $-\infty < s \le t < \infty$, where D_r is the derivative from the right with respect to t.

If $X(t) = X(t,0,\phi)$ is a solution of (12), then

$$|X(t)| \le |a(t)| + \frac{4}{\alpha} e^{\int_0^t p(s)ds} \left[V(0,\phi) + \int_0^t |a(u)| e^{-\int_0^u p(s)ds} du \right],$$

where $V(0,\phi) = \int_{-\alpha}^0 \int_u^0 |g(0,s)||\phi(s)|dsdu$.

Proof. By (12), we easily have

$$|X(t)| \le |a(t)| + \int_{t-\alpha}^{t} |g(t,s)||X(s)|ds. \tag{13}$$

Define

$$V(t,\phi) = \int_{-\alpha}^{0} \int_{u}^{0} |g(t,t+s)||\phi(s)|dsdu, \text{or}$$
$$V(t,X_t) = \int_{-\alpha}^{0} \int_{t+u}^{t} |g(t,s)||X(s)|dsdu.$$

Clearly, $V(t,X_t) \le \alpha \int_{t-\alpha}^t |g(t,s)||X(s)|ds$, which implies

$$-\int_{t-\alpha}^{t} |g(t,s)||X(s)|ds \le -\frac{1}{\alpha} V(t,X_t). \tag{14}$$

Differentiating $V(t,X_t)$ along the solution of (12), we have

$$\begin{aligned} V'_{(12)}(t,X_t) &= \alpha|g(t,t)||X(t)| - \int_{t-\alpha}^{t} |g(t,s)||X(s)|ds \\ &+ \int_{-\alpha}^{0} \int_{t+u}^{t} D_r|g(t,s)||X(s)|dsdu \end{aligned}$$

$$\begin{aligned}
&\leq \alpha|g(t,t)|\left[|a(t)| + \int_{t-\alpha}^{t} |g(t,s)||X(s)|ds\right] \quad \text{(by (13))} \\
&- \int_{t-\alpha}^{t} |g(t,s)||X(s)|ds \\
&= \alpha|g(t,t)||a(t)| + [-1+\alpha|g(t,t)|] \int_{t-\alpha}^{t} |g(t,s)||X(s)|ds \\
&\leq \frac{-1+\alpha|g(t,t)|}{\alpha} V(t,X_t) + \alpha|g(t,t)||a(t)| \quad \text{(by (14))}.
\end{aligned}$$

Therefore

$$V'_{(12)}(t,X_t) - p(t)V(t,X_t) \leq \alpha|g(t,t)||a(t)|.$$

Multiplied by $e^{-\int_0^t p(s)ds}$ and integrated, the last inequality can be written as

$$\begin{aligned}
V(t,X_t) &\leq e^{\int_0^t p(s)ds}\left[V(0,X_0) + \int_0^t \alpha|g(u,u)||a(u)|e^{-\int_0^u p(s)ds}du\right] \\
&\leq e^{\int_0^t p(s)ds}\left[V(0,X_0) + \int_0^t |a(u)|e^{-\int_0^u p(s)ds}du\right]. \qquad (15)
\end{aligned}$$

By changing the order of integration,

$$\begin{aligned}
V(t,X_t) &= \int_{t-\alpha}^{t} |g(t,u)||X(u)|(u-t+\alpha)du \\
&= \int_{t-\alpha}^{t} |g(t,u)||X(u)|udu + (-t+\alpha)\int_{t-\alpha}^{t} |g(t,u)||X(u)|du.
\end{aligned}$$

If $0 \leq t \leq \frac{\alpha}{2}$,

$$V(t,X_t) \geq \frac{\alpha}{2}\int_{t-\alpha}^{t} |g(t,u)||X(u)|du.$$

Then for $0 \leq t \leq \frac{\alpha}{2}$,

$$\begin{aligned}
|X(t)| &\leq |a(t)| + \int_{t-\alpha}^{t} |g(t,u)||X(u)|du \\
&\leq |a(t)| + \frac{2}{\alpha}V(t,X_t) \\
&\leq |a(t)| + \frac{2}{\alpha}e^{\int_0^t p(s)ds}\left[V(0,X_0) + \int_0^t |a(u)|e^{-\int_0^u p(s)ds}du\right] \quad \text{(by (15))}.
\end{aligned}$$

If $t > \frac{\alpha}{2}$,

$$\begin{aligned}
V(t,X_t) &= \int_{t-\alpha}^{t} |g(t,u)||X(u)|(u-t+\alpha)du \\
&\geq \frac{\alpha}{2}\int_{t-\frac{\alpha}{2}}^{t} |g(t,u)||X(u)|du.
\end{aligned}$$

Therefore
$$V(t, X_t) + V(t - \frac{\alpha}{2}, X_{t-\frac{\alpha}{2}}) \geq \frac{\alpha}{2} \int_{t-\alpha}^{t} |g(t,u)||X(u)|du.$$
So for $t > \frac{\alpha}{2}$,
$$\begin{aligned}
|X(t)| &\leq |a(t)| + \int_{t-\alpha}^{t} |g(t,u)||X(u)|du \\
&\leq |a(t)| + \frac{2}{\alpha}\left[V(t, X_t) + V(t - \frac{\alpha}{2}, X_{t-\frac{\alpha}{2}})\right] \\
&\leq |a(t)| + \frac{2}{\alpha} e^{\int_0^t p(s)ds}\left[V(0, X_0) + \int_0^t |a(u)| e^{-\int_0^u p(s)ds} du\right] \\
&+ \frac{2}{\alpha} e^{\int_0^{t-\frac{\alpha}{2}} p(s)ds}\left[V(0, X_0) + \int_0^{t-\frac{\alpha}{2}} |a(u)| e^{-\int_0^u p(s)ds} du\right] \quad \text{(by (15))} \\
&\leq |a(t)| + \frac{4}{\alpha} e^{\int_0^t p(s)ds}\left[V(0, X_0) + \int_0^t |a(u)| e^{-\int_0^u p(s)ds} du\right].
\end{aligned}$$
Therefore, for $t \geq 0$,
$$|X(t)| \leq |a(t)| + \frac{4}{\alpha} e^{\int_0^t p(s)ds}\left[V(0, X_0) + \int_0^t |a(u)| e^{-\int_0^u p(s)ds} du\right].$$

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